

Padé-Improved Extraction of $\alpha_s(M_\tau)$ from R_τ

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Abstract

The perturbative series used to extract $\alpha_s(M_\tau)$ from the τ hadronic width exhibits slow convergence. Asymptotic Padé-approximant and Padé summation techniques provide an estimate of these unknown higher-order effects, leading to values for $\alpha_s(M_\tau)$ that are about 10% smaller than current estimates. Such a reduction improves the agreement of $\alpha_s(M_\tau)$ with the QCD evolution of the strong coupling constant from $\alpha_s(M_Z)$.

The Particle Data Group (PDG) quotes the following values for the strong coupling constant as determined from Z^0 and τ decays [1].

$$\alpha_s(M_\tau) = 0.35 \pm 0.03 \quad (1)$$

$$\alpha_s(M_Z) = 0.119 \pm 0.002 \quad (2)$$

Since these determinations of α_s occur at such widely separated energies, the compatibility of these values of α_s with the QCD evolution of the coupling constant is an important test of both QCD and the phenomenological results used to extract the coupling constant from the experimental data. In particular, $\alpha_s(M_\tau)$ is sufficiently large that presently unknown terms from higher order perturbation theory could substantially alter the value of $\alpha_s(M_\tau)$ extracted from the experimental data. Padé approximant methods provide estimates of the aggregate effect of (presently unknown) terms from higher-order perturbation theory [2, 3, 4, 5]. As shown below, the use of Padé summation to estimate such terms leads to a decrease in the value of $\alpha_s(M_\tau)$ extracted from τ decays, improving the compatibility of $\alpha_s(M_\tau)$ and $\alpha_s(M_Z)$ with the QCD evolution of the coupling constant.

The QCD evolution of the coupling constant is governed by the β function which is now known to 4-loop order [6]. Using the conventions of [7], $a \equiv \frac{\alpha_s}{\pi}$ satisfies the differential equation

$$\mu^2 \frac{da}{d\mu^2} = \beta(a) = -a^2 \sum_{i=0}^{\infty} \beta_i a^i \quad , \quad a \equiv \frac{\alpha_s}{\pi} \quad (3)$$

$$\beta_0 = \frac{11 - \frac{2}{3}n_f}{4} \quad , \quad \beta_1 = \frac{102 - \frac{38}{3}n_f}{16} \quad , \quad \beta_2 = \frac{\frac{2857}{2} - \frac{5033}{18}n_f + \frac{325}{54}n_f^2}{64} \quad (4)$$

$$\beta_3 = 114.23033 - 27.133944n_f + 1.5823791n_f^2 - 5.8566958 \times 10^{-3}n_f^3 \quad (5)$$

Using the value $\alpha_s(M_Z)$ as an initial condition, the coupling constant can be evolved to the desired energy using the differential equation (3). The only subtlety in this approach is the location of flavour thresholds where the number of effective flavour degrees of freedom n_f change. In general, matching conditions must be imposed at these thresholds to relate QCD with n_f quarks to an effective theory with $n_f - 1$ light quarks and a decoupled heavy quark [8]. Using the matching threshold μ_{th} defined by $m_q(\mu_{th}) = \mu_{th}$, where m_q is the running quark mass, the matching condition to three-loop order is [9]

$$a^{(n_f-1)}(\mu_{th}) = a^{(n_f)}(\mu_{th}) \left[1 + 0.1528 \left[a^{(n_f)}(\mu_{th}) \right]^2 + \{0.9721 - 0.0847(n_f - 1)\} \left[a^{(n_f)}(\mu_{th}) \right]^3 \right] \quad (6)$$

leading to a discontinuity of α_s across the threshold. Thus to determine the coupling constant at energies between the c quark threshold and the b quark threshold, the β function with $n_f = 5$ is used to run $\alpha_s^{(5)}$ from M_Z to $\mu_{th} = m_b(\mu_{th}) \equiv m_b$ using (2) as an initial condition. The matching condition (6) is then imposed to find the value of $\alpha_s^{(4)}(m_b)$ which is then used as an initial condition to evolve α_s to lower energies via the $n_f = 4$ beta function.

If $\alpha_s(M_Z)$ is used as the input value to determine the QCD prediction of $\alpha_s(M_\tau)$, then one might legitimately be concerned about the effect of (unknown) higher-order terms in the β function at lower energies where α_s is larger. Padé approximations have proven their utility in determining higher-order terms in the β function. For example, using as input the four-loop β function in N -component massive ϕ^4 scalar field theory [10], asymptotic Padé methods described in Section II of [4] are able to predict the five-loop term to better than 10% of the known five-loop contributions for $N \leq 4$ [3, 11, 12]. When these same methods are applied to QCD, the following predictions for the unknown five-loop contribution to the β function are obtained [11].

$$n_f = 4 : \quad \beta_4 = 83.7563 \quad (7)$$

$$n_f = 5 : \quad \beta_4 = 134.56 \quad (8)$$

From these predictions, β functions containing [2/2] Padé approximants can be constructed to estimate the sum of all higher-order contributions. These Padé-summations, whose Maclaurin expansions reproduce β_1 , β_2 , β_3 and the asymptotic Padé-approximant estimates (7,8) of β_4 , are given by:

$$n_f = 4 : \quad \beta(a) = -\frac{25x^2}{12} \left[\frac{1 - 5.8963a - 4.0110a^2}{1 - 7.4363a + 4.3932a^2} \right] \quad (9)$$

$$n_f = 5 : \quad \beta(a) = -\frac{23x^2}{12} \left[\frac{1 - 5.9761a - 6.9861a^2}{1 - 7.2369a - 0.66390a^2} \right] \quad (10)$$

Thus the QCD prediction of $\alpha_s(M_\tau)$ depends on only two parameters: the initial condition $\alpha_s(M_Z)$ and the position of the five-flavour threshold defined by $m_b(m_b) = m_b$. As will be discussed below, the uncertainty in the Particle Data Group value [1] for this threshold

$$4.1 \text{ GeV} \leq m_b(m_b) \leq 4.4 \text{ GeV} \quad (11)$$

has a negligible effect on the QCD prediction of $\alpha_s(M_\tau)$ compared with the uncertainty in $\alpha_s(M_Z)$ (2).

The compatibility of the experimentally/phenomenologically determined values $\alpha_s(M_Z)$ and $\alpha_s(M_\tau)$ with the QCD evolution of the coupling constant can now be studied. Figure 1 shows the effect on $\alpha_s(Q)$

of progressive increases in the number of perturbative terms in the β function, culminating with the Padé summation (9,10) for β . It is evident that the curves for $\alpha_s(Q)$ appear to converge from below to that generated by the Padé summation of the β function, since the gaps between curves of successive order decrease. Using the input values (2,11) for the QCD evolution of α_s down from the Z^0 to τ mass, we obtain the following range of values for $\alpha_s(M_\tau)$ for successive orders of perturbation theory [11]:

$$2 - \text{loop} \quad 0.3055 \leq \alpha_s(M_\tau) \leq 0.3383 \quad (12)$$

$$3 - \text{loop} \quad 0.3096 \leq \alpha_s(M_\tau) \leq 0.3442 \quad (13)$$

$$4 - \text{loop} \quad 0.3112 \leq \alpha_s(M_\tau) \leq 0.3468 \quad (14)$$

$$\text{Padé summation} \quad 0.3119 \leq \alpha_s(M_\tau) \leq 0.3480 \quad (15)$$

The dominant contribution to the uncertainty (12-15) originates from $\alpha_s(M_Z)$ — the effect of the uncertainty in the five-flavour threshold (11) is inconsequential. It is also evident that Padé-improvement via (9) and (10) does not alter significantly the range of $\alpha_s(M_\tau)$ devolving from the empirical range for $\alpha_s(M_Z)$, as given in (2). Moreover, the ranges (14) and (15) overlap the lower end of the current PDG range (1) for the extraction of $\alpha_s(M_\tau)$ from R_τ . We note, however, that only minimal overlap occurs between (14,15) and the previous (1996) PDG range $\alpha_s(M_\tau) = 0.370 \pm 0.033$ obtained from R_τ [13]. This marginal compatibility with the RG-devolution estimates of $\alpha_s(M_\tau)$ provided the original motivation for us to examine the effect of (estimated) higher-order perturbative corrections on the extraction of $\alpha_s(M_\tau)$ from R_τ .

This extraction occurs by comparing the measured value of R_τ with the calculated value of $\delta^{(0)}$, the purely perturbative QCD corrections to the tree diagram for the calculation of R_τ :

$$R_\tau \equiv \frac{\Gamma(\tau \rightarrow \nu_\tau + \text{hadrons})}{\Gamma(\tau \rightarrow \nu_\tau + e + \bar{\nu}_e)} = 3.058 \left[1 + \delta^{(0)} - (0.014 \pm 0.005) \right] \quad (16)$$

The above expression, as quoted from the current PDG [13], is based on the seminal analysis of Braaten, Narison, and Pich [14], as well as more recent work by Neubert [15]. In particular, the numerical factor in parentheses represents non-perturbative contributions that are dominated by dimension-6 terms. In the $\overline{\text{MS}}$ scheme, the purely perturbative QCD contributions to R_τ are

$$1 + \delta^{(0)} = 1 + a(M_\tau) + 5.2023 [a(M_\tau)]^2 + 26.366 [a(M_\tau)]^3 \quad (17)$$

where $a(M_\tau) \equiv \alpha_s(M_\tau)/\pi$. Using the empirically motivated test value $\alpha_s(M_\tau) = 0.3500$, corresponding to the central value of the PDG range (1), we find that the contributions of successive terms in (17) to $\delta^{(0)}$ are respectively

$$\delta^{(0)} = 0.1114 + 0.06457 + 0.03646 \quad (18)$$

illustrating the slow convergence of perturbative field theory at the mass scale $\mu = M_\tau$. The ratio of successive terms within $\delta^{(0)}$ appears to be nearly 0.6, indicative of significant further contributions from corrections to (17) beyond three-loop-order.

Asymptotic Padé-approximant methods [3, 4] can be utilized to estimate the aggregate effect of higher order terms in the series (17). Given a field-theoretical perturbative series of the form

$$S = 1 + R_1 a + R_2 a^2 + R_3 a^3 + R_4 a^4 \quad (19)$$

in which the coefficients R_k are characterized by asymptotic behaviour $R_k \sim k! C^k k^\gamma$ [16], the coefficient R_4 , which we assume to be unknown, can be estimated from the known terms R_1 , R_2 , and R_3 . An $[N|M]$

Padé-approximant for the series (19) then yields coefficients R_k^{Pade} that differ from those of the series itself via the asymptotic error formula [3, 4]

$$\frac{R_{N+M+1}^{Pade} - R_{N+M+1}}{R_{N+M+1}} \simeq -\frac{M!A^M}{[N+M(1+c)+b]^M} \quad , \quad (20)$$

where $\{A, c, b\}$ are constants to be determined. For example, a $[2|1]$ Padé approximant to (19) would lead to the prediction

$$R_4^{pade} = \frac{R_3^2}{R_2} \quad (21)$$

It has been shown elsewhere [11] that the error formula (20) can be utilized in conjunction with $[0|1]$ and $[1|1]$ approximants to determine A and $(c+b)$, thereby leading to the following “asymptotic Padé-approximant” (APAP) estimate for R_4 :

$$R_4^{APAP} = \frac{R_3^2 [3 + (c+b)]}{R_2 [3 + (c+b) - A]} = \frac{R_3^2 [R_2^3 + R_1 R_2 R_3 - 2 R_1^3 R_3]}{R_2 [2 R_2^3 - R_1^3 R_3 - R_1^2 R_2^2]} \quad (22)$$

If just the $[2|1]$ Padé-approximant is used to estimate [via (21)] the α_s^4 contribution to $\delta^{(0)}$, it is found that [5]

$$1 + \delta^{(0)} = 1 + a(M_\tau) + 5.2023 [a(M_\tau)]^2 + 26.366 [a(M_\tau)]^3 + 109.2 [a(M_\tau)]^4 \quad (23)$$

an estimate very close to that obtained by Kataev and Starshenko using other methods [17]. The APAP-estimate of the α^4 term, obtained via (22), is also positive and somewhat (20%) larger:

$$1 + \delta^{(0)} = 1 + a(M_\tau) + 5.2023 [a(M_\tau)]^2 + 26.366 [a(M_\tau)]^3 + 132.44 [a(M_\tau)]^4 \quad (24)$$

It is significant to note that this prediction is very close to the *maximum* estimated size of the fourth order effect used to determine the theoretical uncertainty in [14], indicating an underestimate of the higher order effects in the extraction of $\alpha_s(M_\tau)$ from R_τ . However, even if $\delta^{(0)}$ includes an estimate of the $[a(M_\tau)]^4$ term, one sees from (24) that the convergence of the perturbative series remains too slow to justify a truncation. For example, if $\alpha_s(M_\tau) = 0.3500$, then the contribution of successive orders to $\delta^{(0)}$ is seen to be

$$\delta^{(0)} = 0.1114 + 0.06457 + 0.03646 + 0.02040 \quad . \quad (25)$$

The (estimated) fourth term is 18% of the leading perturbative contribution. Such slow convergence indicates that further higher order terms should contribute significantly to $\delta^{(0)}$. A Padé-summation, in this case the $[2|2]$ approximant whose first five Maclaurin expansion terms replicate the series (24), provides an estimate of the total effect of higher order terms in a perturbation series [2]. This Padé summation is given by

$$1 + \delta^{(0)} = \frac{1 - 6.5483a(M_\tau) + 10.5030[a(M_\tau)]^2}{1 - 7.5483a(M_\tau) + 12.8514[a(M_\tau)]^2} \quad (26)$$

Figure 2 compares the dependence of $\delta^{(0)}$ on $\alpha_s(M_\tau)$ obtained from (17, 23, 24, 26). These curves correspond respectively to

- **Truncation:** Truncation of contributions to $\delta^{(0)}$ beyond three-loop order (17);
- **[2|1]:** Inclusion via (23) of the [2|1] Padé-approximant estimate of the four-loop contribution to $\delta^{(0)}$;
- **APAP:** Inclusion via (24) of the asymptotic error-formula estimate of the four-loop contribution to $\delta^{(0)}$;
- **PS:** Padé-summation estimate (26) of all higher-loop contributions to $\delta^{(0)}$.

Near the PDG value $\alpha(M_\tau) = 0.350$, the Padé effects lead to a significant *increase* in $\delta^{(0)}$. In comparison with the three-loop perturbative result (17), the size of this enhancement obtained from the Padé summation (26) is roughly twice the enhancement obtained by including Padé estimates of the four-loop contributions (24,23), indicating the significance of the higher-order effects estimated in the [2|2] Padé summation.

In Table 1, we display the values for $\alpha_s(M_\tau)$ one obtains for a given value of $\delta^{(0)}$ by inverting equations (17), (23), (24) or (26). The present empirical range $R_\tau = 3.642 \pm 0.024$ [1], can be used in conjunction with (16) to extract the following range for the purely-perturbative correction $\delta^{(0)}$:

$$\delta^{(0)} = 0.2048 \pm 0.0129 \quad (27)$$

Using Table 1, we find that this empirical range for $\delta^{(0)}$ determines a corresponding range for $\alpha_s(M_\tau)$ for each case listed above:¹

$$\text{Truncation : } \alpha_s(M_\tau) = 0.342 \pm 0.013 \quad (28)$$

$$[2|1] : \alpha_s(M_\tau) = 0.329 \pm 0.011 \quad (29)$$

$$\text{APAP : } \alpha_s(M_\tau) = 0.326 \pm 0.011 \quad (30)$$

$$\text{PS : } \alpha_s(M_\tau) = 0.314 \pm 0.010 \quad (31)$$

It is evident from the above results that progressively sophisticated Padé-estimates of higher-order corrections to $\delta^{(0)}$ lead to progressively lower values for $\alpha_s(M_\tau)$.² In particular, the ranges (29) and (30) are fully enclosed within the ranges (14) and (15) from RG-devolution from $\alpha_s(M_Z)$. The range (31), which we regard as the most accurate reflection of cumulative higher order corrections, is almost entirely enclosed by the lower portion of these RG-ranges. By contrast, only the lower half of the PDG “truncation range” (1) for $\alpha_s(M_\tau)$ is in agreement with the RG-ranges (14) and (15), although the range quoted in (28) does somewhat better than this.

We therefore conclude that higher-order corrections to $\delta^{(0)}$ appear to lower the value of $\alpha_s(M_\tau)$ extracted from R_τ by approximately 10%, but that this lowering seems to improve the overall compatibility of $\alpha_s(M_\tau)$ with the QCD evolution of α_s from the present empirical range for $\alpha_s(M_Z)$. Alternatively, one can conclude that the theoretical uncertainty in $\alpha_s(M_\tau)$ associated with truncation of contributions to $\delta^{(0)}$ past three-loop order is not likely to be bi-directional, as indicated in [1], but is rather an $\mathcal{O}(10\%)$ effect in the downward direction.

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¹The central value for “truncation” increases to $\alpha_s(M_\tau) = 0.347$, consistent with the central value in [1], provided we utilize directly the expressions given in [14] for the non-perturbative contributions, which are weakly α_s dependent.

²A range similar to (29) will also follow from the estimate of the α_s^4 correction to $\delta^{(0)}$ given in [17].

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$\delta^{(0)}$	$\alpha_s(M_\tau)$			
	Truncation	[2 1]	APAP	PS
.190	.3268045079	.3148226450	.3126282933	.3025559995
.191	.3278720443	.3157796995	.3135669239	.3033691708
.192	.3289362783	.3167332605	.3145020527	.3041782424
.193	.3299972341	.3176833554	.3154337071	.3049832461
.194	.3310549354	.3186300106	.3163619139	.3057842132
.195	.3321094053	.3195732518	.3172867002	.3065811741
.196	.3331606671	.3205131058	.3182080919	.3073741596
.197	.3342087438	.3214495976	.3191261155	.3081632002
.198	.3352536583	.3223827527	.3200407965	.3089483254
.199	.3362954327	.3233125966	.3209521606	.3097295647
.200	.3373340894	.3242391534	.3218602326	.3105069480
.201	.3383696501	.3251624483	.3227650377	.3112805039
.202	.3394021367	.3260825052	.3236666006	.3120502613
.203	.3404315704	.3269993482	.3245649460	.3128162486
.204	.3414579730	.3279130010	.3254600973	.3135784937
.205	.3424813650	.3288234873	.3263520787	.3143370249
.206	.3435017672	.3297308298	.3272409138	.3150918692
.207	.3445192005	.3306350518	.3281266258	.3158430544
.208	.3455336849	.3315361762	.3290092380	.3165906074
.209	.3465452410	.3324342253	.3298887731	.3173345551
.210	.3475538883	.3333292210	.3307652536	.3180749239
.211	.3485596462	.3342211861	.3316387021	.3188117395
.212	.3495625353	.3351101415	.3325091400	.3195450287
.213	.3505625736	.3359961093	.3333765900	.3202748172
.214	.3515597814	.3368791108	.3342410733	.3210011296
.215	.3525541766	.3377591664	.3351026111	.3217239918
.216	.3535457785	.3386362978	.3359612251	.3224434288
.217	.3545346055	.3395105254	.3368169358	.3231594650
.218	.3555206760	.3403818694	.3376697639	.3238721252
.219	.3565040080	.3412503501	.3385197303	.3245814337
.220	.3574846194	.3421159877	.3393668551	.3252874143

Table 1: Values of $\alpha_s(M_\tau)$ for given values of $\delta^{(0)}$ obtained by inverting (17) [“Truncation”], (23) [“[2|1]”], (24) [“APAP”], and (26) [“PS”] as discussed in the text.

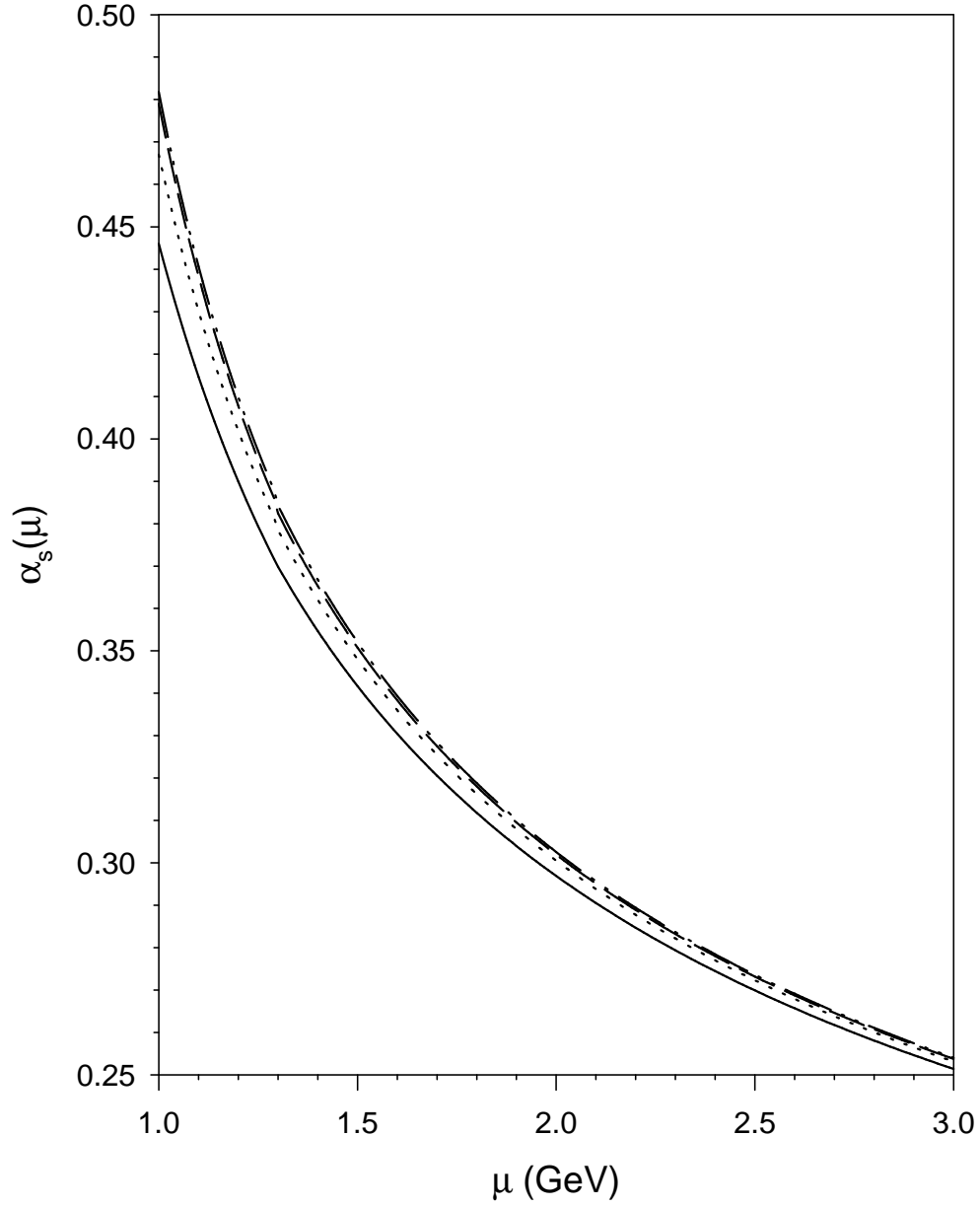


Figure 1: Effect of increasing the order of perturbation theory in the QCD evolution of the strong coupling constant, using $\alpha_s(M_Z)$ as an initial condition. Higher-loop terms in the β function progressively increase α_s from the 2-loop order bottom (solid) curve to the Padé summation top (dashed-dotted) curve, sandwiching the three- and four-loop curves.

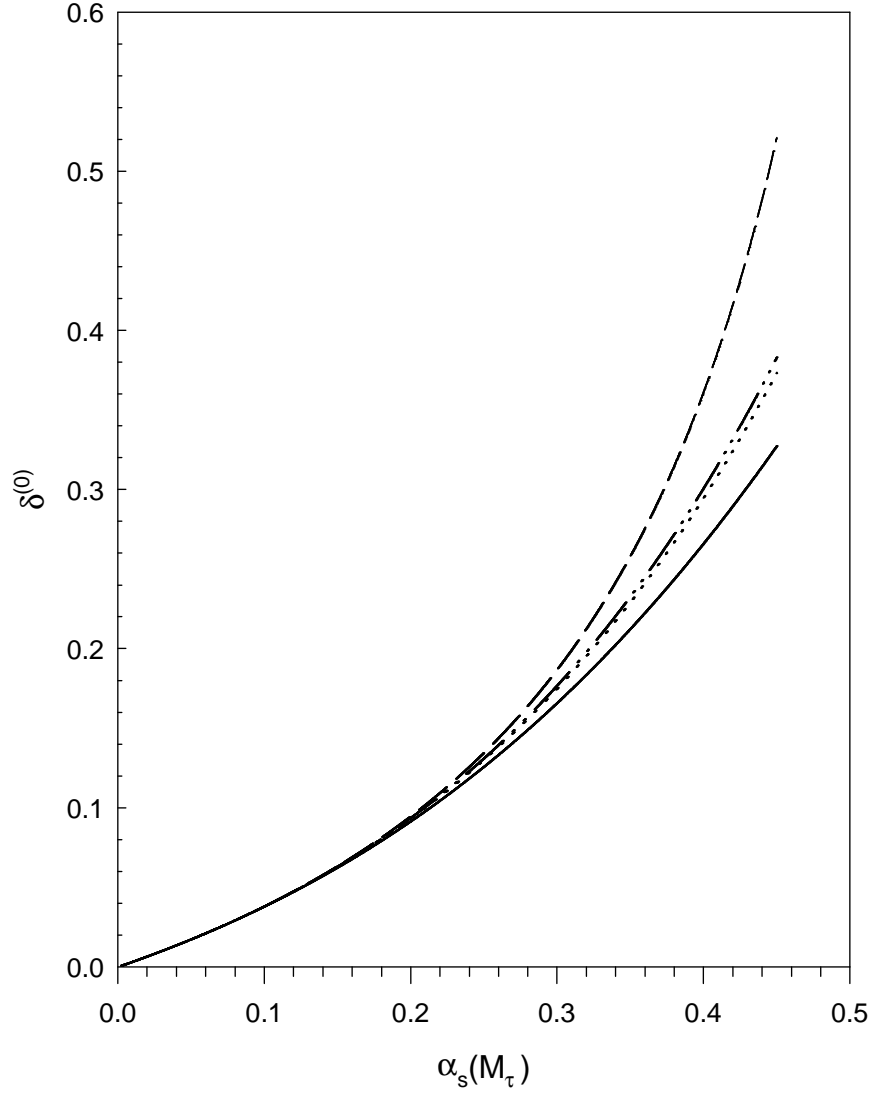


Figure 2: Values for $\delta^{(0)}$ as a function of $\alpha_s(M_\tau)$ using the four different treatments of $\delta^{(0)}$. The solid curve uses (17) [“Truncation”], the dotted curve uses (23) [“[2|1]”], the dashed-dotted curve uses (24) [“APAP”], and the dashed curve uses (26) [“PS”], as discussed in the text.